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BILINEAR OPERATIONS GENERATING ALL OPERATIONS RATIONAL IN A DOMAIN Ω.

BY NORBERT WIENER.

The notion of a corpus, or domain of rationality, is familiar to mathematicians. It can easily be extended to cover sets, not of numbers, but of functions of one or more variables (which may, indeed, degenerate into numbers), exhibiting the group property with reference to the four fundamental operations—addition, subtraction, multiplication, and division—barring only division by 0. Now, it should be noted that these four operations are not only methods of combining functions, but are themselves functions of two variables. Thus $x + y$ is a function of x and y. To combine two functions $f(x)$ and $g(x)$ by addition is to eliminate u and v from the three equations

$$\begin{aligned} z &= u + v, \\ u &= f(x), \\ v &= g(x), \end{aligned}$$

and thus to define z in terms of x.

It is thus possible to regard a function-corpus containing addition, subtraction, multiplication, and division, as a set Σ of functions or operations which contains a sub-set Σ' such that every combination of operations of Σ by operations of Σ' belongs to Σ. In certain cases, Σ' and Σ will coincide. Thus let Σ be the set of all rational functions with coefficients rational in a domain Ω: if $F(x_1, x_2, \cdots, x_n)$, $f_1(y_{11}, y_{12}, \cdots, y_{1k})$, $f_2(y_{21}, y_{22}, \cdots, y_{2l})$, $\cdots, f_n(y_{n1}, y_{n2}, \cdots, y_{nm})$ belong to Σ, $F(f_1, f_2, \cdots, f_n)$ will also belong to Σ, and this whether the y's are all distinct or not. A set of operations, whether a functional corpus or not, which possesses this invariancy under iteration, may be called an *iterative field.*

Iterative fields constitute one of the generalizations of substitution-groups—in fact, substitution-groups are iterative fields of functions of one variable. Now, in a group it is often possible to pick out a very restricted set of elements—a so-called basis—which by their combination generate all the elements of the group. Iterative fields likewise have bases—addition, multiplication, subtraction, and division constitute a basis for the iterative field consisting of all rational operations with rational coefficients. It thus becomes a matter of some interest to determine the smallest possible basis of an iterative field—in particular to determine if and how the field can be generated by the iteration of a single operation. If this is possible, it corresponds to a cyclical group.

In this paper the question at issue is how an iterative field consisting of all rational operations with their coefficients rational in a domain Ω may be generated by the iteration of a single operation of the form

$$x \mid y = \frac{A + Bx + Cy + Dxy}{E + Fx + Gy + Hxy}.$$

The chief result obtained is that if Ω is the domain of rationals, the necessary and sufficient condition that $x \mid y$ constitute a basis is that there is a linear transformation T such that $T^{-1}\{T(x) \mid T(y)\}$ or $T^{-1}\{T(y) \mid T(x)\}$ is of the form

$$\frac{x - y}{x + Ay},$$

where A is any rational number, or else of the form

$$\frac{n(x - y + xy)}{ny + xy},$$

when n is any integer other than 0. Thus $(x - y)/(3x + 6y)$ and $(xy + 3x - 3y - 2)/(xy + x + 3y + 3)$ both serve as bases for all rational operations with rational coefficients, since the first is of the form

$$\frac{1}{3}\left(\frac{3x - 3y}{3x + 2(3y)}\right),$$

and is a transform of

$$\frac{x - y}{x + 2y};$$

while the second may be written

$$\frac{2((x + 1) - (y + 1) + (x + 1)(y + 1))}{2(y + 1) + (x + 1)(y + 1)} - 1,$$

and is a transform of

$$\frac{2(x - y + xy)}{2y + xy}$$

1. **Definitions.** A *bilinear* operation is a binary operation of the form

$$\frac{A + Bx + Cy + Dxy}{E + Fx + Gy + Hxy}.$$

An operation *rational in a domain* Ω is a rational operation whose coefficients belong to Ω.

An operation $f(x_1, x_2, \cdots, x_k)$ is said to *generate* a class K of operations and numbers which may be considered as operations on no variables, when it has the following four properties.

(1) K contains f.

(2) If K contains $g(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_{j-1}, x_j, x_{j+1}, \cdots, x_k)$, K contains $g(x_1, x_2, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_{j-1}, x_i, x_{j+1}, \cdots, x_k)$, and also $g(x_1, x_2, \cdots, x_{i-1}, x_j, x_{i+1}, \cdots, x_{j-1}, x_i, x_{j+1}, \cdots, x_k)$.

(3) If K contains $g(x_1, x_2, \cdots, x_k)$ and $h(y_1, y_2, \cdots, y_l)$, K contains the operation on $x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k, y_1, y_2, \cdots, y_l$ obtained by substituting h for x_i.

(4) K contains no proper sub-set K' with properties (1), (2), and (3).

Infinity will be considered a proper argument for an operation, and $f(\infty)$ will be defined as $\lim_{x \doteq \infty} f(x)$, if this latter expression has a unique significance. Infinity will be considered as a member of every domain.

If T is the non-singular linear transformation

$$x' = \frac{a + bx}{c + dx},$$

the *transform* of $f(x_1, x_2, \cdots, x_k)$ by T is defined as the operation which expresses y in terms of $x_1, x_2, \cdots, x_k$, and is expressed in implicit form in the following set of simultaneous equations.

$$y' = f(x_1', x_2', \cdots, x_k'),$$

$$y' = \frac{a + by}{c + dy},$$

$$x_i' = \frac{a + bx_i}{c + dx_i} \qquad (1 \leq i \leq k).$$

2. **Theorem.*** *Every transform of* $(x - y)/(x + Gy)$ *by a* T *rational in* $\Omega_{(G)}$ *generates all operations rational in* $\Omega_{(G)}$ *if only* $G \neq -1$.

Proof. For the sake of brevity, we shall represent the operation $(x - y)/(x + Gy)$ by the symbol $x|y$. If we can show that whenever $G \neq -1$, we can build up a chain of definitions wherein addition, subtraction, multiplication, division, and G are derived from iterations of $x|y$, it is clear that every operation rational in $\Omega_{(G)}$ can be derived from $x|y$ by iteration. The formation of *ad hoc* definitions to cover all cases where the iterations given become indeterminate offers no difficulty.

There are two cases. First, let $G \neq \pm 1$. Our chain of definitions will read as follows.

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* In this and the following theorems, the operations generated by the bilinear operations in question may be undefined for a set of isolated values of their arguments.

$$\begin{aligned}
0 &= x \mid x, \\
1 &= x \mid 0, \\
x/y &= 1 \mid (y \mid x), \\
xy &= x/(1/y), \\
A(x) &= 1/[(x \mid 1) \mid 1] \text{ [in ordinary symbolism, } A(x) = (G + 1 - x], \\
x - y &= \{xA[(y/x)A(0)]\}/A(0), \\
x + y &= x - [(a - y) - a], \\
G &= [a - 1/(0 \mid x)] - a.
\end{aligned}$$

If $G = 1$, this chain is replaced by the following sequence of definitions:

$$\begin{aligned}
0 &= x \mid x, \\
1 &= x \mid 0, \\
-1 &= 0 \mid x, \\
x/y &= 1 \mid (y \mid x), \\
xy &= x/(1/y), \\
A(x) &= x^2\{[x \mid (1 \mid x)] \mid [(x \mid 1) \cdot (x^2 \mid 1)]\} \text{ [i.e., } 1/(2x - 1)], \\
B(x) &= x^2\{\{(-1)A\{(1 \mid x^2)[(x \mid 1) \mid (-1 \mid x)]\}\} \mid A\{(x \mid 1) \mid (-1 \mid x)\}\} \\
&\qquad \left[\text{i.e., } 1 - \frac{x}{y}\right], \\
x - y &= xB(y/x), \\
x + y &= x - (-1)y.
\end{aligned}$$

Since $x \mid y$ is rational in $\Omega_{(G)}$, it only generates operations rational in $\Omega_{(G)}$, while we have just seen that it generates all operations rational in $\Omega_{(G)}$. A transformation T rational in $\Omega_{(G)}$ is simply a permutation of the numbers and operations of $\Omega_{(G)}$ among themselves. Hence the T-transform of $x \mid y$ generates all operations rational in $\Omega_{(G)}$, and no others.

3. **Theorem.** *Every transform of*

$$\frac{n(x - y + xy)}{ny + xy}$$

by a T *rational in* $\Omega_{(1)}$ *generates all operations rational in* $\Omega_{(1)}$, *if* n *is a non-0 integer in* $\Omega_{(1)}$.

Proof. The discussion of the last theorem applies *in toto*, except that the chair of definitions must be altered. We shall denote

$$\frac{n(x - y + xy)}{ny + xy}$$

by $x \mid y$. Three cases arise.

(1) $n = 1$. Consider the following definitions:

$$\begin{aligned}
0 &= (x \mid x) \mid x, \\
-1 &= 0 \mid x, \\
\infty &= x \mid 0, \\
-x &= \infty \mid (x \mid -1),
\end{aligned}$$

$$\begin{aligned}
A(x, y) &= -\{-(-x|-x)|-[(-y|-y)|(-y|-y)]\}\left[\text{i.e., } 1-\frac{x}{y}\right],\\
1 &= \infty\,|\,\infty,\\
x/y &= A\{A(x, y), 1\},\\
xy &= x/(1/y),\\
x-y &= xA(y, x),\\
x+y &= x-[(a-y)-a].
\end{aligned}$$

(2) $n > 1$. Consider the following definitions:

$$\begin{aligned}
a^1(x) &= x|x,\\
a^{k+1}(x) &= a^1[a^k(x)] \text{ [i.e., } a^k(x) = nx/(kx+n)],\\
0 &= a^n(x)|x,\\
-1 &= 0|x,\\
\infty &= x|0,\\
P(x, y) &= a^{n-1}(x)|a^{n-1}(y) \text{ [i.e., } (x-y+xy)/(y+xy)],\\
-x &= \infty|(x|-1),\\
A(x, y) &= -P\{-P(-x, -x), -P[P(-y, -y), P(-y, -y)]\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{[as before]},\\
1 &= P(\infty, \infty),\\
x/y &= A\{A(x, y), 1\},\\
xy &= x/(1|y),\\
x-y &= xA(y, x),\\
x+y &= x-[(a-y)-a].
\end{aligned}$$

(3) n is negative. $x|y$ is a transform by $y' = y/(y+1)$ of $-n(y-x+xy)/(-nx+xy)$, which differs from the cases considered in (1) and (2) merely by the interchange of x and y.

4. **Theorem.** *If $x|y$ is a bilinear operation generating every operation rational in any domain, it is a transform by a linear transformation rational in some $\Omega_{(G)}$ either of*

$$\frac{x-y}{x+\alpha y} \textit{ or of } \frac{y-x}{y+\alpha x},$$

where α is a number in $\Omega_{(G)}$ other than ∞ or -1, or of

$$\frac{n(x-y+xy)}{ny+xy},$$

where n is a number in $\Omega_{(G)}$ other than ∞ or 0.

Proof. Let

$$x|y = \frac{A+Bx+Cy+Dxy}{E+Fx+Gy+Hxy}.$$

Consider the roots of the cubic equation $x|x = x$. If for any of these roots, r_1, r_2, or r_3, $r_i|r_i$ is determinate, then no sequence of iterations of $|$ on r_i can generate anything but r_i. As the condition we have supposed is equivalent to the assumption that $x|y$ be continuous when x and y are in the neighborhood of r_i, it follows that when x and y are in the neighborhood of r_i, $x|y$ is also in that neighborhood. It results that if $f(x_1, x_2, \cdots, x_n)$ is an operation derived by the iteration of $x|y$, and if $x_1, x_2, \cdots, x_n$ are all in the neighborhood of r_i, $f(x_1, x_2, \cdots, x_n)$ will also be in that neighborhood. This restricts f to a relatively narrow range of variation. In particular, it cannot be any constant other than r_i itself. Since the four fundamental operations $+$, $-$, $\times$, and $\div$ generate all rational constants, it follows that $x|y$ cannot generate these, and hence cannot generate all rational operations with rational coefficients.

Consequently every root of $x|x = x$, be it finite or infinite, is also a finite or infinite root of both the quadratics,

$$Hx^2 + (F + G)x + E = 0, \tag{1}$$

and

$$Dx^2 + (B + C)x + A = 0. \tag{2}$$

If these two equations are not equivalent, the algorithm of the greatest common factor leads to the common root

$$r = -\frac{\begin{vmatrix} E & H \\ A & D \end{vmatrix}}{\begin{vmatrix} F + G & H \\ B + C & D \end{vmatrix}}.$$

This is rational in the domain of the coefficients, and is furthermore, from what has been said, the only finite or infinite root of $x|x = x$. This we shall call case A.

If (1) and (2) are equivalent, it may be seen on inspection that each of their roots is also a root of $x|x = x$, which is *in extenso*

$$Hx^3 + (F + G - D)x^2 + (E - B - C)x - A = 0. \tag{3}$$

This will then have the root $r_1 = D/H$, which is rational in the domain of the coefficients. It follows from what has been said that r_1 is a root of (1). The other root of (1) is $r_2 = (E/H)/(D/H)$, or E/D, which is also rational in the domain of the coefficients. The roots of (3) are then the double root r_1 and the single root r_2. This we shall call case B.

We have seen that a linear transform of an operation has essentially the same iterative properties as its original. Accordingly we may subject the $x|y$ of case A to the transformation $u' = u + r$, thus obtaining the operation

$$x\|y = \frac{A' + B'x + C'y + D'xy}{E' + F'x + G'y + H'xy}.$$

We shall now have 0 as a root of the quadratic

$$(1') \qquad H'x^2 + (F' + G')x + E',$$

and a triple root of

$$(3') \qquad H'x^3 + (F' + G' - D')x^2 + CE' - B' - c')x - A' = 0.$$

It follows that $A' = E' = B' + C' = F' + G' - D' = 0$, whence

$$x \| y = \frac{B'x - B'y + (F' + G')xy}{F'x + G'y + H'xy}.$$

The transformation $u' = B'u/[F'u + (F' + G')]$ reduces this to the form*

$$x \||| y = \frac{n(x - y + xy)}{ny + xy}.$$

Exceptional cases might seem to arise where $B' = 0$ or $F' + G' = 0$. If $B' = 0$, on transforming $x \| y$ by $u' = 1/u$, we get

$$x \||| y = G'x + F'y + H',$$

and it is manifest that the iteration of this can only lead to linear operations. If $F' + G' = 0$, $x \| y$ reduces to the form

$$\frac{x - y}{P(x - y) + Qxy}.$$

If we transform this by $u' = u/(Pu + Q)$ we get

$$x \||| y = \frac{x - y}{xy}.$$

Be it noted that the degree of every term in the numerator is odd, while the power of every term in the denominator is even. It is easy to show that if f and g be two rational operations each in the form either of the quotient of a polynomial whose terms are all of odd degree by a polynomial whose terms are all of even degree, or of the quotient of a polynomial whose terms are all of even degree by a polynomial whose terms are all of odd degree, the operation obtained by substituting f as one of the arguments of g will share in this property. Thus if $f(x, y) = (x - y)/xy$ and $g(x, y, z) = (x^2 + y^2 + z^2)/(x + yz^2)$, $g(x, y, f(u, v))$ will be $(x^4y^2 + x^2y^4 + x^2 - 2xy + y^2)/(x^3y^2 + x^2y - 2xy^2 + y^3)$, where all the terms of the numerator are of even degree and all the terms in the denominator are of odd degree. Hence the iteration of an operation in this class can never lead beyond the confines of the class, and must consequently fail to generate all operations rational in any domain.

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* $n \neq 0$, as the expression would otherwise reduce to a constant. $n \neq \infty$; otherwise, whatever x, $x \||| x = x$.

In case B, submit $x|y$ to the transformation $u' = (r_2u - r_1)/(u - 1)$. $x|y$ will go into an operation

$$x\|y = \frac{A' + B'x + C'y + D'xy}{E' + F'x + G'y + H'xy}.$$

The double root of $x\|x = x$, or of

$$H'x^3 + (F' + G' - D')x^2 + (E' - B' - C')x - A' = 0,$$

will be 0, while the single root will be ∞. Hence

$$A' = H' = E' - B' - C' = 0.$$

Since 0 and ∞ are also roots of

$$H'x^2 + (F' + G')x + E' = 0,$$

and of

$$D'x^2 + (B' + C')x + A' = 0,$$

D' and E' also vanish. From these facts concerning the coefficients, it may be seen that $x\|y$ reduces to

$$\frac{x - y}{Mx + Ny}.$$

Transforming this operation by $u' = u/M$, the result is of the form

$$\frac{x - y}{x + \alpha y}.$$

If $M = 0$, the above transformation is impossible, but if we transform by $u' = -u/N$, we get $(y - x)/y$, of the form $(y - x)/(y + \alpha x)$.

We have thus actually carried through the transformations which reduce $x|y$ to one of the standard forms, and have found them to be rational in the domain of the coefficients of $x|y$.

5. **Theorem.** *If a transform of* $n(x - y + xy)/(ny + xy)$ *generates all operations rational in any domain, then* n *is of the form* p/q, *where* p *is an integer in* $\Omega_{(n)}$ *and* q *is a natural number sharing with* p *no numerical factor, while there is some natural number* k *such that* q *is a factor of* p^k. (An example of the situation contemplated in this theorem is $p = 1 + \sqrt{-5}$; $q = 2$; $k = 2$; $p^k = -4 + 2\sqrt{-5} = q(-2 + \sqrt{-5})$.)

Proof. Consider the class of all numbers in $\Omega_{(n)}$ of the form

$$\frac{p^k + aq}{p^k + bq},$$

where q and k are natural numbers, a and b are integers in $\Omega_{(n)}$, and p is an integer in $\Omega_{(n)}$ sharing no numerical factor with q. Suppose that every number in $\Omega_{(n)}$ can be represented in this form. Let r/s be any number in

$\Omega_{(n)}$ and let r and s be integers in $\Omega_{(n)}$. Then

$$(r - s)p^k = qt,$$

where t is an integer in $\Omega_{(n)}$, namely $sa - rb$. By properly choosing r and s, $r - s$ may be made to contain no factor, actual or ideal, in common with q. Hence, if all numbers in $\Omega_{(n)}$ are reducible to the above form, for some value of k, q is a factor of p^k.

Now, if $n = p/q$, and

$$x|y = \frac{n(x - y + xy)}{ny + xy}$$

generates all operations rational in $\Omega_{(n)}$, $(p^k + aq)/(p^k + bq)$ must assume all values in $\Omega_{(n)}$. For if we let x and y be $(p^k + aq)/(p^k + bq)$ and $(p^l + \alpha q)/(p^l + \beta q)$, it follows that

$$\begin{aligned} x|y &= \frac{(p/q)(x - y + xy)}{(p/q)y + xy} \\ &= \frac{\frac{p}{q}\left\{\frac{p^k + aq}{p^k + bq} - \frac{p^l + \alpha q}{p^l + \beta q} + \frac{p^k + aq}{p^k + bq} \cdot \frac{p^l + \alpha q}{p^l + \beta q}\right\}}{\frac{p}{q} \cdot \frac{p^l + \alpha q}{p^l + \beta q} + \frac{p^k + aq}{p^k + bq} \cdot \frac{p^l + \alpha q}{p^l + \beta q}} \\ &= \frac{p^{k+l+1} + Aq}{p^{k+l+1} + Bq}, \end{aligned}$$

where A and B are integers in $\Omega_{(n)}$. Hence if K is the class of all members of $\Omega_{(n)}$ of the form $(p^k + aq)/(p^k + bq)$, when x and y belong to this class, $x|y$ will also belong to this class. If $x|y$ is to generate all operations rational in $\Omega_{(n)}$, and *a fortiori* all numbers in $\Omega_{(n)}$, K must coincide with $\Omega_{(n)}$. In the previous paragraph we have seen this condition to be equivalent to that expressed in the formulation of the theorem.

6. **Theorem.** *The necessary and sufficient condition that the set of all operations generated by the rational bilinear operation $x|y$ be the set of all rational operations with rational coefficients is that $x|y$ be a rational transform of*

$$\frac{x - y}{x + Ay} \quad \textit{or} \quad \frac{y - x}{y + Ax},$$

where A is any rational number other than -1, *or of*

$$\frac{n(x - y + xy)}{ny + xy},$$

where n is any integer other than 0.

Proof. This results immediately from 2–5, if the domain in question be made that of the rationals.

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